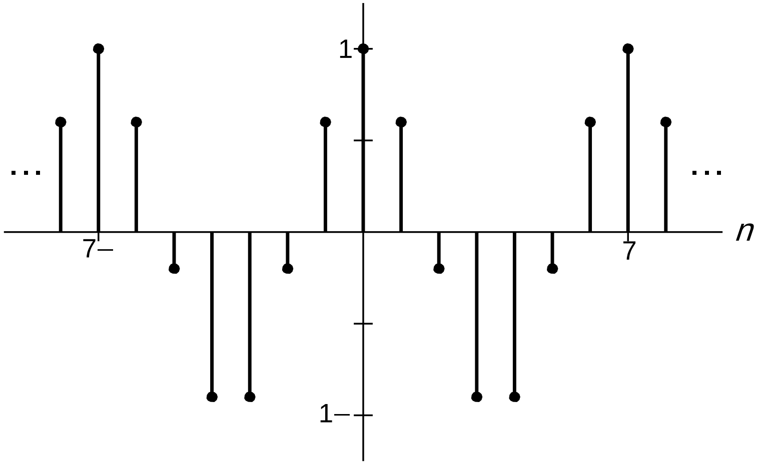
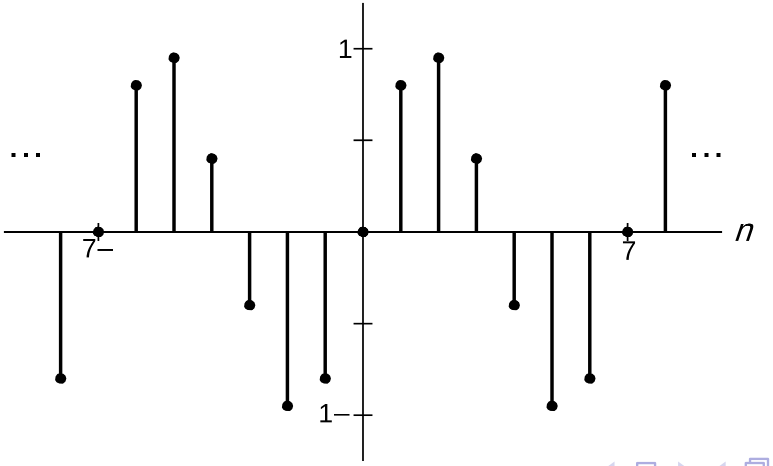
- For $x(n) = e^{j(2\pi/7)n}$, the graphs of $\operatorname{Re}\{x\}$ and $\operatorname{Im}\{x\}$ are shown below.

$\operatorname{Re}\{e^{j(2\pi/7)n}\} = \cos\left(\frac{2\pi}{7}n\right)$



$\operatorname{Im}\{e^{j(2\pi/7)n}\} = \sin\left(\frac{2\pi}{7}n\right)$

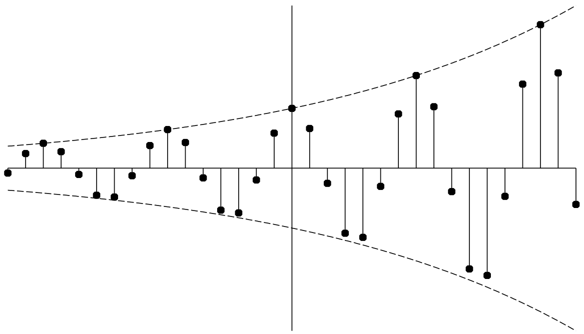


- In the most general case of a complex exponential $x(n) = ca^n$, c and a are both *complex*.
- Letting $c = |c|e^{j\theta}$ and $a = |a|e^{j\Omega}$ where θ and Ω are real, and using Euler's relation, we can rewrite $x(n)$ as

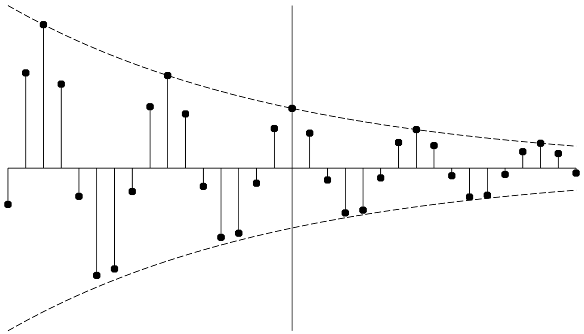
$$x(n) = \underbrace{|c||a|^n\cos(\Omega n + \theta)}_{\mathrm{Re}\{x(n)\}} + j\underbrace{|c||a|^n\sin(\Omega n + \theta)}_{\mathrm{Im}\{x(n)\}}.$$

- Thus, $\mathrm{Re}\{x\}$ and $\mathrm{Im}\{x\}$ are each the product of a real exponential and real sinusoid.
- One of *several distinct modes* of behavior is exhibited by x, depending on the value of a.
- If $|a| = 1$, $\mathrm{Re}\{x\}$ and $\mathrm{Im}\{x\}$ are *real sinusoids*.
- If $|a| > 1$, $\mathrm{Re}\{x\}$ and $\mathrm{Im}\{x\}$ are each the *product of a real sinusoid and a growing real exponential*.
- If $|a| < 1$, $\mathrm{Re}\{x\}$ and $\mathrm{Im}\{x\}$ are each the *product of a real sinusoid and a decaying real exponential*.

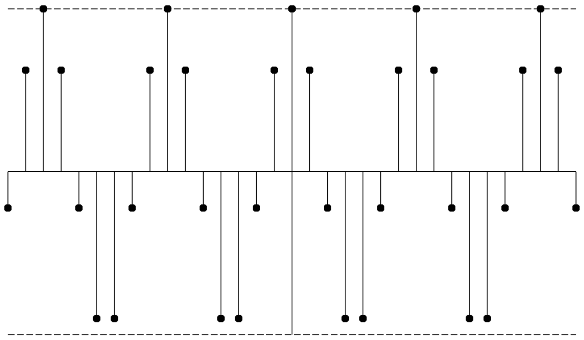
- The *various modes of behavior* for $\mathrm{Re}\{x\}$ and $\mathrm{Im}\{x\}$ are illustrated below.



$|a| > 1$



$|a| < 1$



$|a| = 1$

Relationship Between Complex Exponentials and Real Sinusoids

- From Euler's relation, a complex sinusoid can be expressed as the sum of two real sinusoids as

$$ce^{j\Omega n} = c\cos\Omega n + jc\sin\Omega n.$$

- Moreover, a real sinusoid can be expressed as the sum of two complex sinusoids using the identities

$$c\cos(\Omega n + \theta) = \frac{c}{2}\left[e^{j(\Omega n + \theta)} + e^{-j(\Omega n + \theta)}\right] \quad \text{and}$$

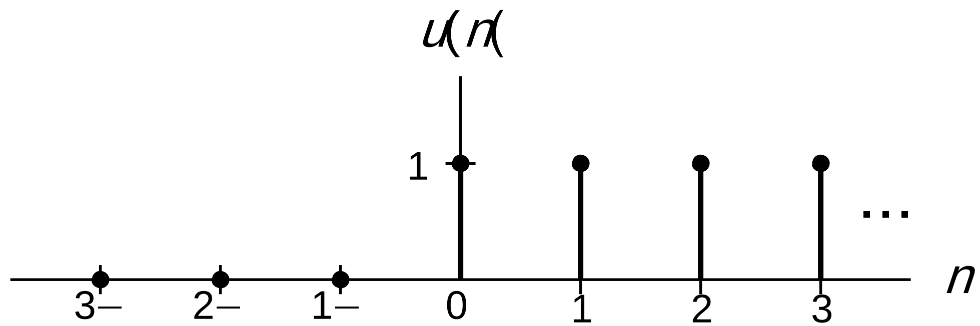
$$c\sin(\Omega n + \theta) = \frac{c}{2j}\left[e^{j(\Omega n + \theta)} - e^{-j(\Omega n + \theta)}\right].$$

- Note that, above, we are simply *restating results* from the (appendix) material on complex analysis.

- The unit-step sequence, denoted u, is defined as

$$u(n) = \begin{cases} 1 & \text{if } n \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

- A plot of this sequence is shown below.



- A unit rectangular pulse is a sequence of the form

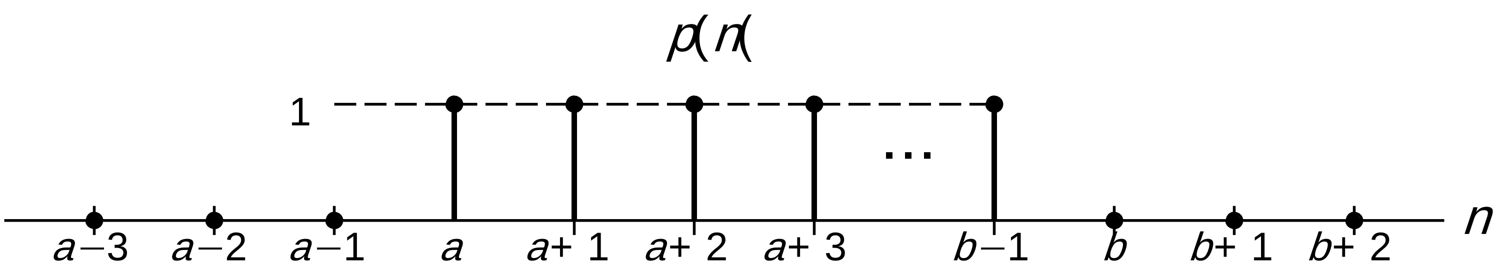
$$p(n) = \begin{cases} 1 & \text{if } a \leq n < b \\ 0 & \text{otherwise} \end{cases}$$

  where a and b are integer constants satisfying $a < b$.

- Such a sequence can be expressed in terms of the unit-step sequence as

$$p(n) = u(n-a) - u(n-b).$$

- The graph of a unit rectangular pulse has the general form shown below.



- The unit-impulse sequence (also known as the delta sequence), denoted δ, is defined as

$$\delta(n) = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{otherwise.} \end{cases}$$

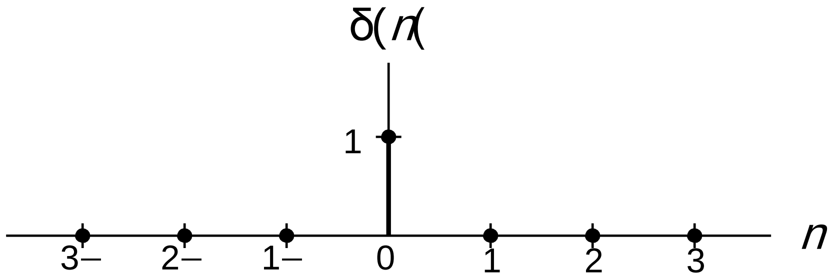
- The first-order difference of u is δ. That is,

$$\delta(n) = u(n) - u(n-1).$$

- The running sum of δ is u. That is,

$$u(n) = \sum_{k=-\infty}^{n} \delta(k).$$

- A plot of δ is shown below.



- For any sequence x and any integer constant n_0, the following identity holds:

$$x(n)\delta(n-n_0) = x(n_0)\delta(n-n_0).$$

- For any sequence x and any integer constant n_0, the following identity holds:

$$\sum_{n=-\infty}^{\infty} x(n)\delta(n-n_0) = x(n_0).$$

- Trivially, the sequence δ is also even.

Section 7.4

Discrete-Time (DT) Systems

- A system with input x and output y can be described by the equation

$$y = \mathcal{H}\{x\},$$

where $\mathcal{H}$ denotes an operator (i.e., transformation).

- Note that the operator $\mathcal{H}$ *maps a function to a function* (not a number to a number).
- Alternatively, we can express the above relationship using the notation

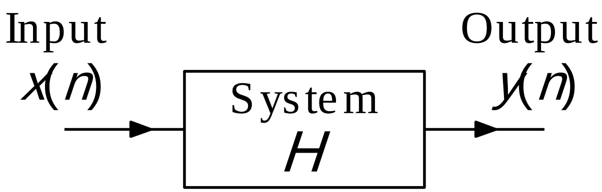
$$x \xrightarrow{\mathcal{H}} y.$$

- If clear from the context, the operator $\mathcal{H}$ is often omitted, yielding the abbreviated notation

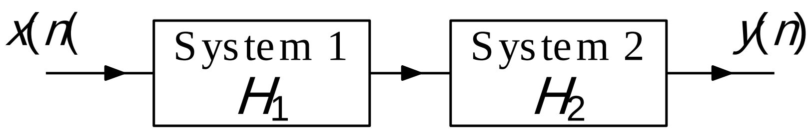
$$x \to y.$$

- Note that the symbols "$\to$" and "=" have *very different* meanings.
- The symbol "$\to$" should be read as "*produces*" (not as "equals").

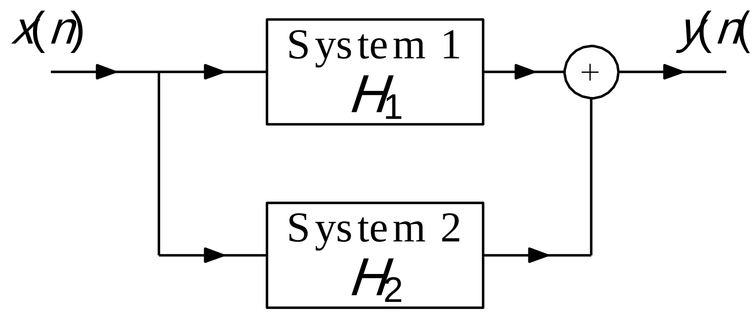
- Often, a system defined by the operator H and having the input x and output y is represented in the form of a *block diagram* as shown below.



- *Two basic ways* in which systems can be interconnected are shown below.



Series



Parallel

- A series (or cascade) connection ties the output of one system to the input of the other.
- The overall series-connected system is described by the equation

$$y = H_2\{H_1\{x\}\}$$

- A parallel connection ties the inputs of both systems together and sums their outputs.
- The overall parallel-connected system is described by the equation

$$y = H_1\{x\} + H_2\{x\}$$

Section 7.5

Properties of (DT) Systems

- A system with input x and output y is said to have memory if, for any integer n_0, $y(n_0)$ depends on $x(n)$ for some $n \neq n_0$.
- A system that does not have memory is said to be memoryless.
- Although simple, a memoryless system is *not very flexible*, since its current output value cannot rely on past or future values of the input.
- A system with input x and output y is said to be causal if, for every integer n_0, $y(n_0)$ does not depend on $x(n)$ for some $n > n_0$.
- If the independent variable n represents time, a system must be causal in order to be *physically realizable*.
- Noncausal systems can sometimes be useful in practice, however, since the independent variable *need not always represent time*. For example, in some situations, the independent variable might represent position.

- The inverse of a system H is another system H^{-1} such that the combined effect of H cascaded with H^{-1} is a system where the input and output are equal.
- A system is said to be invertible if it has a corresponding inverse system (i.e., its inverse exists).
- Equivalently, a system is invertible if its input x can always be *uniquely* determined from its output y.
- Note that the invertibility of a system (which involves mappings between *functions*) and the invertibility of a function (which involves mappings between *numbers*) are *fundamentally different* things.
- An invertible system will always produce *distinct outputs* from any two *distinct inputs*.
- To show that a system is *invertible*, we simply find the *inverse system*. To
- show that a system is *not invertible*, we find *two distinct inputs* that result in *identical outputs*.
- In practical terms, invertible systems are "nice" in the sense that their *effects can be undone*.